

# An Algorithm for Computing the Limit Points of the Quasi-component of a Regular Chain

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## Abstract

For a regular chain  $R$ , we propose an algorithm which computes the (non-trivial) limit points of the quasi-component of  $R$ , that is, the set  $\overline{W(R)} \setminus W(R)$ . Our procedure relies on Puiseux series expansions and does not require to compute a system of generators of the saturated ideal of  $R$ . We focus on the case where this saturated ideal has dimension one and we discuss extensions of this work in higher dimensions. We provide experimental results illustrating the benefits of our algorithms.

## 1 Introduction

The theory of regular chains, since its introduction by J.F. Ritt [26], has been applied successfully in many areas including parametric algebraic systems [10], differential systems [11, 3, 17], difference systems [16], intersection multiplicity [21], unmixed decompositions [18] and primary decomposition [27] of polynomial ideals, cylindrical algebraic decomposition [9], parametric [31] and non-parametric [5] semi-algebraic systems. Today, regular chains are at the core of algorithms for triangular decomposition of polynomial systems, which are available in several software packages [19, 29, 30]. Moreover, these algorithms provide back-engines for computer algebra system front-end solvers, such as MAPLE's `solve` command.

One of the algorithmic strengths of the theory of regular chains is its *regularity test* procedure. Given a polynomial  $p$  and a regular chain  $R$ , both in a multivariate polynomial ring  $\mathbf{k}[X_1, \dots, X_n]$  over a field  $\mathbf{k}$ , this procedure computes regular chains  $R_1, \dots, R_e$  such that  $R_1, \dots, R_e$  is a decomposition of  $R$  in some technical sense<sup>1</sup> and for each  $1 \leq i \leq e$  the polynomial  $p$  is either null or regular modulo the saturated ideal of  $R_i$ . Thanks to the D5 Principle [13], this regularity test avoids factorization into irreducible polynomials and involves only polynomial GCD and resultant computations.

One of the technical difficulties of this theory, however, is the fact that regular chains do not fit well in the “usual algebraic-geometric dictionary” (Chapter 4, [12]). Indeed, the “good” zero set encoded by a regular chain  $R$  is a constructible set  $W(R)$ , called the *quasi-component* of  $R$ , which does not correspond exactly to the “good” ideal encoded by  $R$ , namely  $\text{sat}(R)$ , the *saturated ideal* of  $R$ . In fact, the affine variety defined by  $\text{sat}(R)$  equals  $\overline{W(R)}$ , that is, the Zariski closure of  $W(R)$ .

For this reason, a decomposition algorithm, such as the one of M. Kalkbrener [18] (which, for an input polynomial ideal  $\mathcal{I}$  computes regular chains  $R_1, \dots, R_e$  such that  $\sqrt{\mathcal{I}}$  equals the intersection of the radicals of the saturated ideals of  $R_1, \dots, R_e$ ) can not be seen as a decomposition algorithm for the variety  $V(\mathcal{I})$ . Indeed, the output of Kalkbrener's algorithm yields  $V(\mathcal{I}) = \overline{W(R_1)} \cup \dots \cup \overline{W(R_e)}$  while a decomposition of the form  $V(\mathcal{I}) = W(R_1) \cup \dots \cup W(R_f)$  would be more explicit.

<sup>1</sup>The radical of the saturated ideal of  $R$  is equal to the intersection of the radicals of the saturated ideals of  $R_1, \dots, R_e$ .

Kalkbrener’s decompositions, and in fact all decompositions of differential ideals [11, 3, 17] raise another notorious issue: the *Ritt problem*, stated as follows. Given two regular chains (algebraic or differential)  $R$  and  $S$ , check whether the inclusion of saturated ideals  $\text{sat}(R) \subseteq \text{sat}(S)$  holds or not. In the algebraic case, this inclusion can be tested by computing a set of generators of  $\text{sat}(R)$ , using Gröbner bases. In practice, this solution is too expensive for the purpose of removing redundant components in Kalkbrener’s decompositions and only some criteria are applied [20]. In the differential case, there has not even an algorithmic solution.

In the algebraic case, both issues would be resolved if one would have a practically efficient procedure with the following specification: for the regular chain  $R$  compute regular chains  $R_1, \dots, R_e$  such that we have  $\overline{W(R)} = W(R_1) \cup \dots \cup W(R_e)$ . If in addition, such procedure does not require a system of generators of  $\text{sat}(R)$ , this might suggest a solution in the differential version of the Ritt problem.

In this paper, we propose a solution to this algorithmic quest, in the algebraic case. To be precise, our procedure computes the *non-trivial limit points* of the quasi-component  $W(R)$ , that is, the set  $\lim(W(R)) := \overline{W(R)} \setminus W(R)$ . This turns out to be  $\overline{W(R)} \cap V(h_R)$ , where  $V(h_R)$  is the hypersurface defined by the product of the initials of  $R$ . We focus on the case where the saturated ideal of  $R$  has dimension one. In Section 10, we sketch a solution in higher dimension.

When the regular chain  $R$  consists of a single polynomial  $r$ , primitive w.r.t. its main variable, one can easily check that  $\lim(W(R)) = V(r, h_R)$  holds. Unfortunately, there is no generalization of this result when  $R$  consists of several polynomials, unless  $R$  enjoys remarkable properties, such as being a *primitive regular chain* [20]. To overcome this difficulty, it becomes necessary to view  $R$  as a “parametric representation” of the quasi-component  $W(R)$ . In this setting, the points of  $\lim(W(R))$  can be computed as limits (in the usual sense of the Euclidean topology<sup>2</sup>) of sequences of points along “branches” (in the sense of the theory of algebraic curves) of  $W(R)$ . It turns out that these limits can be obtained as constant terms of convergent Puiseux series defining the “branches” of  $W(R)$  in the neighborhood of the points of interest.

Here comes the main technical difficulty of this approach. When computing a particular point of  $\lim(W(R))$ , one needs to follow one branch per defining equation of  $R$ . Following a branch means computing a truncated Puiseux expansion about a point. Since the equation of  $R$  defining a given variable, say  $X_j$ , depends on the equations of  $R$  defining the variables  $X_{j-1}, X_{j-2}, \dots$ , the truncated Puiseux expansion for  $X_j$  is defined by an equation whose coefficients involve the truncated Puiseux expansions for  $X_{j-1}, X_{j-2}, \dots$ .

From Sections 4 to 8, we show that this principle indeed computes the desired limit points. In particular, we introduce the notion of a *system of Puiseux parametrizations of a regular chain*, see Section 4. This allows to state in Theorem 3 a concise formula for  $\lim(W(R))$  in terms of this latter notion. Then, we estimate to which accuracy one needs to effectively compute such a system of Puiseux parametrizations in order to deduce  $\lim(W(R))$ , see Theorem 6 in Section 7.

In Section 9, we report on a preliminary implementation of the algorithms presented in this paper. We evaluate our code by applying it to the question of removing redundant components in Kalkbrener’s decompositions and observe the benefits of this strategy.

In order to facilitate the presentation of those technical materials, we dedicate Section 3 to the case of regular chains in 3 variables. Section 2 briefly reviews notions from the theories

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<sup>2</sup>This identification of the closures of  $W(R)$  in Zariski topology and the Euclidean topology holds when  $\mathbf{k}$  is  $\mathbb{C}$ .

of regular chains and algebraic curves. We conclude this introduction with a detailed example.

Consider the regular chain  $R = \{r_1, r_2\} \subset \mathbf{k}[X_1, X_2, X_3]$  with  $r_1 = X_1 X_2^2 + X_2 + 1$ ,  $r_2 = (X_1 + 2)X_1 X_3^2 + (X_2 + 1)(X_3 + 1)$ . Then, we have  $h_R = X_1(X_1 + 2)$ . To determine  $\lim(R)$ , we need to compute Puiseux series expansions of  $r_1$  about  $X_1 = 0$  and  $X_1 = -2$ . We start with  $X_1 = 0$ . The two Puiseux expansions of  $r_1$  about  $X_1 = 0$  are:

$$\begin{aligned} [X_1 = T, X_2 = \frac{-T^2 - T}{T} + O(T^2)], \\ [X_1 = T, X_2 = \frac{-1 + T^2 + T}{T} + O(T^2)]. \end{aligned}$$

The second expansion does not result in a new limit point. After, substituting the first expansion into  $r_2$ , we have:

$$\begin{aligned} r'_2 &= r_2(X_1 = T, X_2 = \frac{-T^2 - T}{T} + O(T^2), X_3) \\ &= T((T + 2)X_3^2 + (O(T^3) - 5T^2 - 2T - 1 + 1)(X_3 + 1)). \end{aligned}$$

Now, we compute Puiseux series expansions of  $r'_2$  which are

$$\begin{aligned} [T = T, X_3 = 1 + T + O(T^2)], \\ [T = T, X_3 = -1/2 - 1/4T + O(T^2)]. \end{aligned}$$

So the regular chains  $\{X_1, X_2 + 1, X_3 - 1\}$  and  $\{X_1, X_2 + 1, X_3 + 1/2\}$  give the limit points of  $W(R)$  about  $X_1 = 0$ .

Next, we consider  $X_1 = -2$ . We compute Puiseux series expansions of  $r_1$  about the point  $X_1 = -2$ . We have:

$$\begin{aligned} [X_1 = T - 2, X_2 = 1 + 1/3T + O(T^2)], \\ [X_1 = T - 2, X_2 = -1/2 - 1/12T + O(T^2)]. \end{aligned}$$

After substitution into  $r_2$ , we obtain:

$$\begin{aligned} r'_{12} &= r_2(X_1 = T - 2, X_2 = 1 + 1/3T + O(T^2), X_3) \\ &= (T - 2)TX_3^2 + (2 + 1/3T + O(T^2))(X_3 + 1) \\ r'_{22} &= r_2([X_1 = T - 2, X_2 = -1/2 - 1/12T + O(T^2)]) \\ &= (T - 2)TX_3^2 + (1/2 - 1/12T + O(T^2))(X_3 + 1). \end{aligned}$$

So those Puiseux expansions of  $r'_{12}$  and  $r'_{22}$  about  $T = 0$  which result in a limit point are as follows:

- i) for  $r'_{12}$ :  $[T = T, X_3 = \frac{T^2 - T}{T} + O(T^2)]$
- ii) for  $r'_{22}$ :  $[T = T, X_3 = \frac{4T^2 - T}{T} + O(T^2)]$

Thus, the limit points of  $R$  about the point  $X_1 = -2$  can be represented by the regular chains  $\{X_1 + 2, X_2 - 1, X_3 + 1\}$  and  $\{X_1 + 2, X_2 + 1/2, X_3 + 1\}$ .

One can check that a triangular decomposition of the system  $R \cup \{X_1\}$  is  $\{X_2 + 1, X_1\}$  and, thus, does not yield  $\lim(W(R)) \cap V(X_1)$ , but in fact a superset of it.

## 2 Preliminaries

This section is a brief review of various notions from the theories of regular chains, algebraic curves and topology. For these latter subjects, our references are the textbooks of R.J. Walker [28], G. Fischer [15] and J. R. Munkres [24]. The notations and hypotheses introduced in this section are used throughout the sequel of the paper.

**Multivariate polynomials.** Let  $\mathbf{k}$  be a field which is algebraically closed. Let  $X_1 < \dots < X_s$  be  $s \geq 1$  ordered variables. We denote by  $\mathbf{k}[X_1, \dots, X_s]$  the ring of polynomials in the variables  $X_1, \dots, X_s$  and with coefficients in  $\mathbf{k}$ . For a non-constant polynomial  $p \in \mathbf{k}[X_1, \dots, X_s]$ , the greatest variable in  $p$  is called *main variable* of  $p$ , denoted by  $\text{mvar}(p)$ , and the leading coefficient of  $p$  w.r.t.  $\text{mvar}(p)$  is called *initial* of  $p$ , denoted by  $\text{init}(p)$ .

**Zariski topology.** We denote by  $\mathbb{A}^s$  the *affine  $s$ -space* over  $\mathbf{k}$ . An *affine variety* of  $\mathbb{A}^s$  is the set of common zeroes of a collection  $F \subseteq \mathbf{k}[X_1, \dots, X_s]$  of polynomials. The *Zariski topology* on  $\mathbb{A}^s$  is the topology whose closed sets are the affine varieties of  $\mathbb{A}^s$ . The *Zariski closure* of a subset  $W \subseteq \mathbb{A}^s$  is the intersection of all affine varieties containing  $W$ . This is also the set of common zeroes of the polynomials in  $\mathbf{k}[X_1, \dots, X_s]$  vanishing at any point of  $W$ .

**Relation between Zariski topology and the Euclidean topology.** When  $\mathbf{k} = \mathbb{C}$ , the affine space  $\mathbb{A}^s$  is endowed with both Zariski topology and the Euclidean topology. The basic open sets of the Euclidean topology are the balls while the basic open sets of Zariski topology are the complements of hypersurfaces. A Zariski closed (resp. open) set is closed (resp. open) in the Euclidean topology on  $\mathbb{A}^s$ . The following properties emphasize the fact that Zariski topology is coarser than the Euclidean topology: every nonempty Euclidean open set is Zariski dense and every nonempty Zariski open set is dense in the Euclidean topology on  $\mathbb{A}^s$ . However, the closures of a constructible set in Zariski topology and the Euclidean topology are equal. More formally, we have the following (Corollary 1 in I.10 of [23]) key result. Let  $V \subseteq \mathbb{A}^s$  be an irreducible affine variety and  $U \subseteq V$  be open in the Zariski topology induced on  $V$ . Then the closure of  $U$  in Zariski topology and the closure of  $U$  in the Euclidean topology are both equal to  $V$ .

**Limit points.** Let  $(X, \tau)$  be a topological space. A point  $p \in X$  is a *limit* of a sequence  $(x_n, n \in \mathbb{N})$  of points of  $X$  if, for every neighborhood  $U$  of  $p$ , there exists an  $N$  such that, for every  $n \geq N$ , we have  $x_n \in U$ ; when this holds we write  $\lim_{n \rightarrow \infty} x_n = p$ . If  $X$  is a Hausdorff space then limits of sequences are unique, when they exist. Let  $S \subseteq X$  be a subset. A point  $p \in X$  is a *limit point* of  $S$  if every neighborhood of  $p$  contains at least one point of  $S$  different from  $p$  itself. Equivalently,  $p$  is a limit point of  $S$  if it is in the closure of  $S \setminus \{p\}$ . In addition, the closure of  $S$  is equal to the union of  $S$  and the set of its limit points. If the space  $X$  is sequential, and in particular if  $X$  is a metric space, the point  $p$  is a limit point of  $S$  if and only if there exists a sequence  $(x_n, n \in \mathbb{N})$  of points of  $S \setminus \{p\}$  with  $p$  as limit. In practice, the “interesting” limit points of  $S$  are those which do not belong to  $S$ . For this reason, we call such limit points *non-trivial* and we denote by  $\lim(S)$  the set of non-trivial limit points of  $S$ .

**Regular chain.** A set  $R$  of non-constant polynomials in  $\mathbf{k}[X_1, \dots, X_s]$  is called a *triangular set*, if for all  $p, q \in R$  with  $p \neq q$  we have  $\text{mvar}(p) \neq \text{mvar}(q)$ . For a nonempty triangular set  $R$ , we define the *saturated ideal*  $\text{sat}(R)$  of  $R$  to be the ideal  $\langle R \rangle : h_R^\infty$ , where  $h_R$  is the product of the initials of the polynomials in  $R$ . The empty set is also regarded as a triangular set, whose saturated ideal is the trivial ideal  $\langle 0 \rangle$ . From now on,  $R$  denotes a triangular set of  $\mathbf{k}[X_1, \dots, X_s]$ . The ideal  $\text{sat}(R)$  has several properties, in particular it is unmixed [4]. We

denote its height by  $e$ , thus  $\text{sat}(R)$  has dimension  $s-e$ . Without loss of generality, we assume that  $\mathbf{k}[X_1, \dots, X_{s-e}] \cap \text{sat}(R)$  is the trivial ideal  $\langle 0 \rangle$ . For all  $1 \leq i \leq e$ , we denote by  $r_i$  the polynomial of  $R$  whose main variable is  $X_{i+s-e}$  and by  $h_i$  the initial of  $r_i$ . Thus  $h_R$  is the product  $h_1 \cdots h_e$ . We say that  $R$  is a *regular chain* whenever  $R$  is empty or  $\{r_1, \dots, r_{e-1}\}$  is a regular chain and  $h_e$  is regular modulo the saturated ideal  $\text{sat}(\{r_1, \dots, r_{e-1}\})$ . The regular chain  $R$  is said *strongly normalized* whenever  $h_R \in \mathbf{k}[X_1, \dots, X_{s-e}]$  holds. If  $R$  is not strongly normalized, one can compute a regular chain  $N$  which is strongly normalized and such that  $\text{sat}(R) = \text{sat}(N)$  and  $V(h_N) = V(\widehat{h_R})$  both hold, where  $\widehat{h_R}$  is the iterated resultant of  $h_R$  w.r.t  $R$ . See [7].

**Limit points of the quasi-component of a regular chain.** We denote by  $W(R) := V(R) \setminus V(h_R)$  the *quasi-component* of  $R$ , that is, the common zeros of  $R$  that do not cancel  $h_R$ . The above discussion implies that the closure of  $W(R)$  in Zariski topology and the closure of  $W(R)$  in the Euclidean topology are both equal to  $V(\text{sat}(R))$ , that is, the affine variety of  $\text{sat}(R)$ . We denote by  $\overline{W(R)}$  this common closure. We call *limit points* of  $W(R)$  the elements of  $\lim(W(R))$ .

**Rings of formal power series.** Recall that  $\mathbf{k}$  is an algebraically closed field. From now on, we further assume that  $\mathbf{k}$  is topologically complete. Hence  $\mathbf{k}$  may be the field  $\mathbb{C}$  of complex numbers but not the algebraic closure of the field  $\mathbb{Q}$  of rational numbers. We denote by  $\mathbf{k}[[X_1, \dots, X_s]]$  and  $\mathbf{k}\langle X_1, \dots, X_s \rangle$  the rings of formal and convergent power series in  $X_1, \dots, X_s$  with coefficients in  $\mathbf{k}$ . Note that the ring  $\mathbf{k}\langle X_1, \dots, X_s \rangle$  is a subring of  $\mathbf{k}[[X_1, \dots, X_s]]$ . When  $s = 1$ , we write  $T$  instead of  $X_1$ . Thus  $\mathbf{k}[[T]]$  and  $\mathbf{k}\langle T \rangle$  are the rings of formal and convergent univariate power series in  $T$  and coefficients in  $\mathbf{k}$ . For  $f \in \mathbf{k}[[X_1, \dots, X_s]]$ , its *order* is defined by

$$\text{ord}(f) = \begin{cases} \min\{d \mid f_{(d)} \neq 0\} & \text{if } f \neq 0, \\ \infty & \text{if } f = 0. \end{cases}$$

where  $f_{(d)}$  is the *homogeneous part* of  $f$  in degree  $d$ . Recall that  $\mathbf{k}[[X_1, \dots, X_s]]$  is topologically complete for Krull Topology and that  $\mathbf{k}\langle X_1, \dots, X_s \rangle$  is a Banach Algebra for the norm defined by  $\|f\|_\rho = \sum_e |a_e| \rho^e$  where  $f = \sum_e a_e X^e \in \mathbf{k}[[X_1, \dots, X_s]]$  and  $\rho = (\rho_1, \dots, \rho_s) \in \mathbb{R}_{>0}^s$ . We denote by  $\mathcal{M}_s$  the only maximal ideal of  $\mathbf{k}[[X_1, \dots, X_s]]$ , that is,

$$\mathcal{M}_s = \{f \in \mathbf{k}[[X_1, \dots, X_s]] \mid \text{ord}(f) \geq 1\}.$$

Let  $f \in \mathbf{k}[[X_1, \dots, X_s]]$  with  $f \neq 0$ . Let  $k \in \mathbb{N}$ . We say that  $f$  is (1) *general* in  $X_s$  if  $f \neq 0 \pmod{\mathcal{M}_{s-1}}$ , (2) *general* in  $X_s$  of order  $k$  if we have  $\text{ord}(f \pmod{\mathcal{M}_{s-1}}) = k$ .

**Formal Puiseux series.** We denote by  $\mathbf{k}[[T^*]] = \bigcup_{n=1}^{\infty} \mathbf{k}[[T^{\frac{1}{n}}]]$  the ring of *formal Puiseux series*. For a fixed  $\varphi \in \mathbf{k}[[T^*]]$ , there is an  $n \in \mathbb{N}_{>0}$  such that  $\varphi \in \mathbf{k}[[T^{\frac{1}{n}}]]$ . Hence  $\varphi = \sum_{m=0}^{\infty} a_m T^{\frac{m}{n}}$ , where  $a_m \in \mathbf{k}$ . We call *order of*  $\varphi$  the rational number defined by  $\text{ord}(\varphi) = \min\{\frac{m}{n} \mid a_m \neq 0\} \geq 0$ . We denote by  $\mathbf{k}((T^*))$  the quotient field of  $\mathbf{k}[[T^*]]$ .

**Convergent Puiseux series.** Let  $\varphi \in \mathbb{C}[[T^*]]$  and  $n \in \mathbb{N}$  such that  $\varphi = f(T^{\frac{1}{n}})$  with  $f \in \mathbb{C}[[T]]$  holds. We say that the Puiseux series  $\varphi$  is *convergent* if we have  $f \in \mathbb{C}\langle T \rangle$ . Convergent Puiseux series form an integral domain denoted by  $\mathbb{C}\langle T^* \rangle$ ; its quotient field is denoted by  $\mathbb{C}((T^*))$ . For every  $\varphi \in \mathbb{C}((T^*))$ , there exist  $n \in \mathbb{Z}$ ,  $r \in \mathbb{N}_{>0}$  and a sequence of complex numbers  $a_n, a_{n+1}, a_{n+2}, \dots$  such that we have

$$\varphi = \sum_{m=n}^{\infty} a_m T^{\frac{m}{r}} \quad \text{and} \quad a_n \neq 0.$$

Then, we define  $\text{ord}(\varphi) = \frac{n}{r}$ .

**Puiseux Theorem.** If  $\mathbf{k}$  has characteristic zero, the field  $\mathbf{k}(\langle T^* \rangle)$  is the algebraic closure of the field of formal Laurent series over  $\mathbf{k}$ . Moreover, if  $\mathbf{k} = \mathbb{C}$ , the field  $\mathbb{C}(\langle T^* \rangle)$  is algebraically closed as well. From now on, we assume  $\mathbf{k} = \mathbb{C}$ .

**Puiseux expansion.** Let  $\mathbb{B} = \mathbb{C}(\langle X^* \rangle)$  or  $\mathbb{C}(\langle X^* \rangle)$ . Let  $f \in \mathbb{B}[Y]$ , where  $d := \deg(f, Y) > 0$ . Let  $h := \text{lc}(f, Y)$ . According to Puiseux Theorem, there exists  $\varphi_i \in \mathbb{B}$ ,  $i = 1, \dots, d$ , such that  $\frac{f}{h} = (Y - \varphi_1) \cdots (Y - \varphi_d)$ . We call  $\varphi_1, \dots, \varphi_d$  the *Puiseux expansions* of  $f$  at the origin.

**Puiseux parametrization.** Let  $f \in \mathbb{C}\langle X \rangle[Y]$ . A parametrization of  $f$  is a pair  $(\psi(T), \varphi(T))$  of elements of  $\mathbb{C}\langle T \rangle$  for some new variable  $T$ , such that (1)  $f(\psi(T), \varphi(T)) = 0$  holds in  $\mathbb{C}\langle T \rangle$ , (2) we have  $0 < \text{ord}(\psi(T))$ , and (3)  $\psi(T)$  and  $\varphi(T)$  are not both in  $\mathbb{C}$ . The parametrization  $(\psi(T), \varphi(T))$  is *irreducible* if there is no integer  $k > 1$  such that both  $\psi(T)$  and  $\varphi(T)$  are in  $\mathbb{C}\langle T^k \rangle$ . We call an irreducible parametrization  $(\psi(T), \varphi(T))$  of  $f$  a *Puiseux parametrization* of  $f$ , if there exists a positive integer  $\varsigma$  such that  $\psi(T) = T^\varsigma$ . The index  $\varsigma$  is called the *ramification index* of the parametrization  $(T^\varsigma, \varphi(T))$ . It is intrinsic to  $f$  and  $\varsigma \leq \deg(f, Y)$ . Let  $z_1, \dots, z_\varsigma$  denote the primitive roots of unity of order  $\varsigma$  in  $\mathbb{C}$ . Then  $\varphi(z_i X^{1/\varsigma})$ , for  $i = 1, \dots, \varsigma$ , are  $\varsigma$  Puiseux expansions of  $f$ .

We conclude this section by a few lemmas which are immediate consequences of the above review.

**Lemma 1.** We have:  $\lim(W(R)) = \overline{W(R)} \cap V(h_R)$ . In particular,  $\lim(W(R))$  is either empty or an affine variety of dimension  $s - e - 1$ .

**Lemma 2.** If  $R$  is a primitive regular chain, that is, if  $R$  is a system of generators of its saturated ideal, then we have  $\lim(W(R)) = V(R) \cap V(h_R)$ .

**Lemma 3.** If  $N$  is a strongly normalized regular chain such that  $\text{sat}(R) = \text{sat}(N)$  and  $V(h_N) = V(\widehat{h_R})$  both hold, then we have  $\lim(W(R)) \subseteq \lim(W(N))$ .

**Lemma 4.** Let  $x \in \mathbb{A}^s$  such that  $x \notin W(R)$ . Then  $x \in \lim(W(R))$  holds if and only if there exists a sequence  $(\alpha_n, n \in \mathbb{N})$  of points in  $\mathbb{A}^s$  such that  $\alpha_n \in W(R)$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \alpha_n = x$ .

**Lemma 5.** Recall that  $R$  writes  $\{r_1, \dots, r_e\}$ . If  $e > 1$  holds, writing  $R' = \{r_1, \dots, r_{e-1}\}$  and  $r = r_e$ , we have

$$\lim(W(R' \cup r)) \subseteq \lim(W(R')) \cap \lim(W(r)).$$

**Lemma 6.** Let  $\varphi \in \mathbb{C}(\langle T^* \rangle)$  and let  $p/q \in \mathbb{Q}$  be the order of  $\varphi$ . Let  $(\alpha_n, n \in \mathbb{N})$  be a sequence of complex numbers converging to zero and let  $N$  be a positive integer such that  $(\varphi(\alpha_n), n \geq N)$  is well defined. Then, if  $p/q < 0$  holds, the sequence  $(\varphi(\alpha_n), n \geq N)$  escapes to infinity while if  $p/q \geq 0$ , the sequence  $(\varphi(\alpha_n), n \geq N)$  converges to the complex number  $\varphi(0)$ .

### 3 Basic techniques

This section is an overview of the basic techniques of this paper. This presentation is meant to help the non-expert reader understand our objectives and solutions. In particular, the

results of this section are stated for regular chains in three variables, while the statements of Sections 4 to 8 do not have this restriction.

Recall that  $R \subseteq \mathbb{C}[X_1, \dots, X_s]$  is a regular chain whose saturated ideal has height  $1 \leq e \leq s$ . As mentioned in the introduction, we mainly focus on the case  $e = s - 1$ , that is,  $\text{sat}(R)$  has dimension one.

Lemma 1 and the assumption  $e = s - 1$  imply that  $\lim(W(R))$  consists of finitely many points.

We further assume that  $R$  is strongly normalized, thus we have  $h_R$  lies in  $\mathbb{C}[X_1]$ .

Lemma 2 and the assumption  $h_R \in \mathbb{C}[X_1]$  imply that computing  $\lim(W(R))$  reduces to check, for each root  $\alpha \in \mathbb{C}$  of  $h_R$  whether or not there is a point  $x \in \lim(W(R))$  whose  $X_1$ -coordinate is  $\alpha$ . Without loss of generality, it is enough to develop our results for the case  $\alpha = 0$ . Indeed, a change of coordinates can be used to reduce to this latter assumption.

We start by considering the case  $n = 2$ . Thus, our regular chain  $R$  consists of a single polynomial  $r_1 \in \mathbb{C}[X_1, X_2]$  whose initial  $h_1$  satisfies  $h_1(0) = 0$ . Lemma 7 provides a necessary and sufficient condition for a point of  $(\alpha, \beta) \in \mathbb{A}^2$ , with  $\alpha = 0$ , to satisfy  $(\alpha, \beta) \in \lim(W(\{r_1\}))$ .

Let  $d$  be the degree of  $r_1$  in  $X_2$ . Applying Puiseux Theorem, we consider  $\varphi_1, \dots, \varphi_d \in \mathbb{C}(\langle X_1^* \rangle)$  such that the following holds

$$\frac{r_1}{h_1} = (X_2 - \varphi_1) \cdots (X_2 - \varphi_d) \quad (1)$$

in  $\mathbb{C}(\langle X_1^* \rangle)[X_2]$ . We assume that the series  $\varphi_1, \dots, \varphi_d$  are numbered in such a way that each of  $\varphi_1, \dots, \varphi_c$  has a non-negative order while each of  $\varphi_{c+1}, \dots, \varphi_d$  has a negative order, for some  $c$  such that  $0 \leq c \leq d$ .

**Lemma 7.** *With  $h_1(0) = 0$ , for all  $\beta \in \mathbb{C}$ , the following two conditions are equivalent*

- (i)  $(0, \beta) \in \lim(W(r_1))$  holds,
- (ii) there exists  $1 \leq j \leq c$  and a sequence  $(\alpha_n, n \in \mathbb{N})$  of complex numbers such that the sequence  $(\varphi_j(\alpha_n), n \in \mathbb{N})$  is well defined, we have  $h_1(\alpha_n) \neq 0$  for all  $n \in \mathbb{N}$  and we have

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \varphi_j(\alpha_n) = \beta.$$

*Proof.* We first prove the implication (ii)  $\Rightarrow$  (i). Equation (1) together with (ii) implies  $(\alpha_n, \varphi_j(\alpha_n)) \in V(r_1)$  for all  $n \in \mathbb{N}$ . Since we also have  $(\alpha_n, \varphi_j(\alpha_n)) \notin V(h_1)$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} (\alpha_n, \varphi_j(\alpha_n)) = (0, \beta)$ , we deduce (i), thanks to Lemma 4.

We now prove the implication (i)  $\Rightarrow$  (ii). By Lemma 4, there exists a sequence  $((\alpha_n, \beta_n), n \in \mathbb{N})$  in  $\mathbb{A}^2$  such that for all  $n \in \mathbb{N}$  we have: (1)  $h_1(\alpha_n) \neq 0$ , (2)  $r_1(\alpha_n, \beta_n) = 0$ , and (3)  $\lim_{n \rightarrow \infty} (\alpha_n, \beta_n) = (0, \beta)$ . Since  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , each series  $\varphi_1(\alpha_n), \dots, \varphi_d(\alpha_n)$  is well defined for  $n$  larger than some positive integer  $N$ . Hypotheses (1) and (2), together with Equation (1), imply that for all  $n \geq N$  the product

$$(\beta_n - \varphi_1(\alpha_n)) \cdots (\beta_n - \varphi_c(\alpha_n))(\beta_n - \varphi_{c+1}(\alpha_n)) \cdots (\beta_n - \varphi_d(\alpha_n))$$

is 0. Since  $\lim_{n \rightarrow \infty} \beta_n = \beta$ , and by definition of the integer  $c$ , each of the sequences  $(\beta_n - \varphi_1(\alpha_n)), \dots, (\beta_n - \varphi_c(\alpha_n))$  converges while each of the sequences  $(\beta_n - \varphi_{c+1}(\alpha_n)), \dots, (\beta_n - \varphi_d(\alpha_n))$  escapes to infinity. Thus, for  $n$  large enough the product  $(\beta_n - \varphi_1(\alpha_n)) \cdots (\beta_n - \varphi_c(\alpha_n))$  is zero. Therefore, one of sequences  $(\beta_n - \varphi_1(\alpha_n)), \dots, (\beta_n - \varphi_c(\alpha_n))$  converges to 0 and the conclusion follows.  $\square$

Lemmas 6 and 7 immediately imply the following.

**Proposition 1.** *With  $h_1(0) = 0$ , for all  $\beta \in \mathbb{C}$ , we have*

$$(0, \beta) \in \lim(W(r_1)) \iff \beta \in \{\varphi_1(0), \dots, \varphi_c(0)\}.$$

Next, we consider the case  $n = 3$ . Hence, our regular chain  $R$  consists of two polynomials  $r_1 \in \mathbb{C}[X_1, X_2]$  and  $r_2 \in \mathbb{C}[X_1, X_2, X_3]$  with respective initials  $h_1$  and  $h_2$ . We assume that 0 is a root of the product  $h_1 h_2$  and we are looking for all  $\beta \in \mathbb{C}$  and all  $\gamma \in \mathbb{C}$  such that  $(0, \beta, \gamma) \in \lim(W(r_1, r_2))$ .

Lemma 5 tells us that  $(0, \beta, \gamma) \in \lim(W(r_1, r_2))$  implies  $(0, \beta) \in \lim(W(r_1))$ . This observation together with Proposition 1 yields immediately the following.

**Proposition 2.** *With  $h_1(0) = 0$  and  $h_2(0) \neq 0$ , assuming that  $r_1$  is primitive over  $\mathbb{C}[X_1]$ , for all  $\beta \in \mathbb{C}$  and all  $\gamma \in \mathbb{C}$ , we have*

$$(0, \beta, \gamma) \in \lim(W(r_1, r_2)) \iff (0, \beta, \gamma) \in V(r_1, r_2).$$

We turn now our attention to the case  $h_1(0) = h_2(0) = 0$ . Since  $(0, \beta) \in \lim(W(r_1))$  is a necessary condition for  $(0, \beta, \gamma) \in \lim(W(r_1, r_2))$  to hold we apply Proposition 1 and assume  $\beta \in \{\varphi_1(0), \dots, \varphi_c(0)\}$ . Without loss of generality, we further assume  $\beta = 0$ . For each  $1 \leq j \leq c$ , such that  $\varphi_j(0) = 0$  holds, we define the univariate polynomial  $f_2^j \in \mathbb{C}(\langle X_1^* \rangle)[X_3]$  by

$$f_2^j(X_1, X_3) = r_2(X_1, \varphi_j(X_1), X_3). \quad (2)$$

Let  $b$  be the degree of  $f_2^j$ . Applying again Puiseux theorem, we consider  $\psi_1, \dots, \psi_b \in \mathbb{C}(\langle X_1^* \rangle)$  such that the following holds

$$\frac{f_2^j}{h_2} = (X_3 - \psi_1) \cdots (X_3 - \psi_b) \quad (3)$$

in  $\mathbb{C}(\langle X_1^* \rangle)[X_3]$ . We assume that the series  $\psi_1, \dots, \psi_b$  are numbered in such a way that each of  $\psi_1, \dots, \psi_a$  has a non-negative order while each of  $\psi_{a+1}, \dots, \psi_b$  has a negative order, for some  $a$  such that  $0 \leq a \leq b$ .

**Lemma 8.** *For all  $\gamma \in \mathbb{C}$ , the following two conditions are equivalent.*

- (i)  $(0, 0, \gamma) \in \lim(W(r_1, r_2))$  holds,
- (ii) *there exist integers  $j, k$  with  $1 \leq j \leq c$  and  $1 \leq k \leq a$ , and two sequences  $(\alpha_n, n \in \mathbb{N})$ ,  $(\beta_n, n \in \mathbb{N})$  of complex numbers such that:*
  - (a) *the sequences  $(\varphi_j(\alpha_n), n \in \mathbb{N})$  and  $(\psi_k(\beta_n), n \in \mathbb{N})$  are well defined,*
  - (b)  *$h_1(\alpha_n) \neq 0$  and  $h_2(\alpha_n) \neq 0$ , for all  $n \in \mathbb{N}$ ,*
  - (c)  *$\beta_n = \varphi_j(\alpha_n)$ , for all  $n \in \mathbb{N}$ ,*
  - (d)  *$\lim_{n \rightarrow \infty} (\alpha_n, \beta_n, \psi_k(\beta_n)) = (0, 0, \gamma)$ .*

*Proof.* Proving the implication  $(ii) \Rightarrow (i)$  is easy. We now prove the implication  $(i) \Rightarrow (ii)$ . By Lemma 4, there exists a sequence  $((\alpha_n, \beta_n, \gamma_n), n \in \mathbb{N})$  in  $\mathbb{A}^3$  s.t. for all  $n \in \mathbb{N}$  we have: (1)  $h_1(\alpha_n) \neq 0$ , (2)  $h_2(\alpha_n) \neq 0$ , (3)  $r_1(\alpha_n, \beta_n) = 0$ , (4)  $r_2(\alpha_n, \beta_n, \gamma_n) = 0$ , (5)  $\lim_{n \rightarrow \infty} (\alpha_n, \beta_n, \gamma_n) = (0, 0, \gamma)$ . Following the proof of Lemma 7, we know that for  $n$  large enough the product  $(\beta_n - \varphi_1(\alpha_n)) \cdots (\beta_n - \varphi_c(\alpha_n))$  is zero. Therefore, from one of the



sequences  $(\beta_n - \varphi_1(\alpha_n)), \dots, (\beta_n - \varphi_c(\alpha_n))$ , say the  $j$ -th, one can extract an (infinite) subsequence whose terms are all zero. Thus, without loss of generality, we assume that  $\beta_n = \varphi_j(\alpha_n)$  holds, for all  $n \in \mathbb{N}$ . Hence, for all  $n \in \mathbb{N}$ , we have  $f_2^j(\alpha_n, \gamma_n) = r_2(\alpha_n, \beta_n, \gamma_n) = 0$ . Together with Equation (3) and following the proof of Lemma 7, we deduce the desired result.  $\square$

Lemmas 6 and 8 immediately imply the following.

**Proposition 3.** *For all  $\gamma \in \mathbb{C}$ , the following two conditions are equivalent.*

- (i)  $(0, 0, \gamma) \in \lim(W(r_1, r_2))$  holds,
- (ii) there exist integers  $j, k$  with  $1 \leq j \leq c$  and  $1 \leq k \leq a$ , such that  $\varphi_j(0) = 0$  and  $\psi_k(0) = \gamma$ .

Therefore, applying Puiseux theorem to  $r_1$  and  $f_2^j$ , then checking the constant terms of the series  $\psi_1, \dots, \psi_b$  provides a way to compute all  $\gamma \in \mathbb{C}$  such that  $(0, 0, \gamma)$  is a limit point of  $W(r_1, r_2)$ . Theorem 3 in Sections 4 states this principle formally for an arbitrary regular chain  $R$  in dimension one.

Finally, one should also consider the case  $h_1(0) \neq 0, h_2(0) = 0$ . In fact, it is easy to see that this latter case can be handled in a similar manner as the case  $h_1(0) = 0, h_2(0) = 0$ .

## 4 Puiseux expansions of a regular chain

In this section, we introduce the notion of Puiseux expansions of a regular chain, motivated by the work of [22, 1] on Puiseux expansions of space curves.

**Lemma 9.** *Let  $R = \{r_1, \dots, r_{s-1}\} \subset \mathbb{C}[X_1 < \dots < X_s]$  be a strongly normalized regular chain whose saturated ideal has dimension one. Recall that  $h_R(X_1)$  denotes the product of the initials of polynomials in  $R$ . Let  $\rho > 0$  be small enough such that the set  $0 < |X_1| < \rho$  does not contain any zeros of  $h_R$ . Denote by  $U_\rho := \{x = (x_1, \dots, x_s) \in \mathbb{C}^s \mid 0 < |x_1| < \rho\}$ . Denote by  $V_\rho(R) := V(R) \cap U_\rho$ . Then we have  $W(R) \cap U_\rho = V_\rho(R)$ . Let  $R' := \{\text{primpart}(r_1), \dots, \text{primpart}(r_{s-1})\}$ . Then  $V_\rho(R) = V_\rho(R')$ .*

*Proof.* Let  $x \in W(R) \cap U_\rho$ , then  $x \in V(R)$  and  $x \in U_\rho$  hold, which implies that  $W(R) \cap U_\rho \subseteq V(R) \cap U_\rho$ . Let  $x \in V(R) \cap U_\rho$ . Since  $U_\rho \cap V(h_R) = \emptyset$ , we have  $x \in W(R)$ . Thus  $V(R) \cap U_\rho \subseteq W(R) \cap U_\rho$ . So  $W(R) \cap U_\rho = V_\rho(R)$ . Similarly we have  $V_\rho(R) = V_\rho(R')$ .  $\square$

**Notation 1.** Let  $W \subseteq \mathbb{C}^s$ . Denote  $\lim_0(W) := \{x = (x_1, \dots, x_s) \in \mathbb{C}^s \mid x \in \lim(W) \text{ and } x_1 = 0\}$ .

**Lemma 10.** *Let  $R = \{r_1, \dots, r_{s-1}\} \subset \mathbb{C}[X_1 < \dots < X_s]$ . Then we have  $\lim_0(W(R)) = \lim_0(V_\rho(R))$ .*

*Proof.* By Lemma 9, we have  $W(R) \cap U_\rho(R) = V_\rho(R)$ . On the other hand  $\lim_0(W(R)) = \lim_0(W(R) \cap U_\rho(R))$ . Thus  $\lim_0(W(R)) = \lim_0(V_\rho(R))$  holds.  $\square$

**Lemma 11.** *Let  $R$  be as in Lemma 9. For  $1 \leq i \leq s-1$ , let  $d_i := \deg(r_i, X_{i+1})$ . Then  $R$  generates a zero-dimensional ideal in  $\mathbb{C}(\langle X_1^* \rangle)[X_2, \dots, X_s]$ . Let  $V^*(R)$  be the zero set of  $R$  in  $\mathbb{C}(\langle X_1^* \rangle)^{s-1}$ . Then  $V^*(R)$  has exactly  $\prod_{i=1}^{s-1} d_i$  points, counting multiplicities.*

*Proof.* It follows directly from the definition of regular chain, Bezout bound and the fact that  $\mathbb{C}(\langle X_1^* \rangle)^{s-1}$  is an algebraically closed field.  $\square$

**Definition 1.** We use the notions in Lemma 11. Each point in  $V^*(R)$  is called a Puiseux expansion of  $R$ .

**Notation 2.** Let  $m = |V^*(R)|$ . Write  $V^*(R) = \{\Phi_1, \dots, \Phi_m\}$  with  $\Phi_i = (\Phi_i^1(X_1), \dots, \Phi_i^{s-1}(X_1))$ , for  $i = 1, \dots, m$ . Let  $\rho > 0$  be small enough such that for  $1 \leq i \leq m$ ,  $1 \leq j \leq s-1$ , each  $\Phi_i^j(X_1)$  converges in  $0 < |X_1| < \rho$ . We define  $V_\rho^*(R) := \cup_{i=1}^m \{x \in \mathbb{C}^s \mid 0 < |x_1| < \rho, x_{j+1} = \Phi_i^j(x_1), j = 1, \dots, s-1\}$ .

**Theorem 1.** We have  $V_\rho^*(R) = V_\rho(R)$ .

*Proof.* We prove this by induction on  $s$ . For  $i = 1, \dots, s-1$ , recall that  $h_i$  is the initial of  $r_i$ . If  $s = 2$ , we have

$$r_1(X_1, X_2) = h_1(X_1) \prod_{i=1}^{d_1} (X_2 - \Phi_i^1(X_1)).$$

So  $V_\rho^*(R) = V_\rho(R)$  clearly holds.

Write  $R = R' \cup \{r_{s-1}\}$ ,  $X' = X_2, \dots, X_{s-1}$ ,  $X = (X_1, X', X_s)$ ,  $x' = x_2, \dots, x_{s-1}$ ,  $x = (x_1, x', x_s)$ , and  $m' = |V^*(R')|$ . For  $i = 1, \dots, m$ , let  $\Phi_i = (\Phi'_i, \Phi_i^{s-1})$ , where  $\Phi'_i$  stands for  $\Phi_i^1, \dots, \Phi_i^{s-2}$ . Assume the theorem holds for  $R'$ , that is  $V_\rho^*(R') = V_\rho(R')$ . For any  $i = 1, \dots, m'$ , there exist  $i_k \in \{1, \dots, m\}$ ,  $k = 1, \dots, d_{s-1}$  such that

$$r_{s-1}(X_1, X' = \Phi'_i, X_s) = h_1(X_1) \prod_{k=1}^{d_{s-1}} (X_s - \Phi_{i_k}^{s-1}(X_1)). \quad (4)$$

Note that  $V^*(R) = \cup_{i=1}^{m'} \cup_{k=1}^{d_{s-1}} \{(X' = \Phi'_i, X_s = \Phi_{i_k}^{s-1})\}$ . Therefore, by induction hypothesis and Equation (4), we have

$$\begin{aligned} V_\rho^*(R) &= \cup_{i=1}^{m'} \cup_{k=1}^{d_{s-1}} \{x \mid x \in U_\rho, x' = \Phi'_i(x_1), x_s = \Phi_{i_k}^{s-1}(x_1)\} \\ &= \cup_{k=1}^{d_{s-1}} \{x \mid (x_1, x') \in V_\rho^*(R'), x_s = \Phi_{i_k}^{s-1}(x_1)\} \\ &= \{x \mid (x_1, x') \in V_\rho^*(R'), r_{s-1}(x_1, x', x_s) = 0\} \\ &= \{x \mid (x_1, x') \in V_\rho(R'), r_{s-1}(x_1, x', x_s) = 0\} \\ &= V_\rho(R). \end{aligned}$$

□

**Theorem 2.** Let  $V_{\geq 0}^*(R) := \{\Phi = (\Phi^1, \dots, \Phi^{s-1}) \in V^*(R) \mid \text{ord}(\Phi^j) \geq 0, j = 1, \dots, s-1\}$ . Then we have

$$\lim_0(W(R)) = \cup_{\Phi \in V_{\geq 0}^*(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

*Proof.* By definition of  $V_{\geq 0}^*(R)$ , we immediately have

$$\lim_0(V_\rho^*(R)) = \cup_{\Phi \in V_{\geq 0}^*(R)} \{(X_1 = 0, \Phi(X_1 = 0))\}.$$

Next, by Theorem 1, we have  $V_\rho^*(R) = V_\rho(R)$ . Thus, we have  $\lim_0(V_\rho^*(R)) = \lim_0(V_\rho(R))$ . Besides, with Lemma 10, we have  $\lim_0(W(R)) = \lim_0(V_\rho(R))$ . Thus the theorem holds. □

**Definition 2.** Let  $V_{\geq 0}^*(R) := \{\Phi = (\Phi^1, \dots, \Phi^{s-1}) \in V^*(R) \mid \text{ord}(\Phi^j) \geq 0, j = 1, \dots, s-1\}$ . Let  $M = |V_{\geq 0}^*(R)|$ . For each  $\Phi_i = (\Phi_i^1, \dots, \Phi_i^{s-1}) \in V_{\geq 0}^*(R)$ ,  $1 \leq i \leq M$ , we know that  $\Phi_i^j \in \mathbb{C}(\langle X_1^* \rangle)$ . Moreover, by Equation (4), we know that for  $j = 1, \dots, s-1$ ,  $\Phi_i^j$  is a Puiseux expansion of  $r_j(X_1, X_2 = \Phi_i^1, \dots, X_j = \Phi_i^{j-1}, X_{j+1})$ . Let  $\varsigma_{i,j}$  be the ramification index of  $\Phi_i^j$

and  $(T^{\varsigma_{i,j}}, X_{j+1} = \varphi_i^j(T))$ , where  $\varphi_i^j \in \mathbb{C}\langle T \rangle$ , be the corresponding Puiseux parametrization of  $\Phi_i^j$ . Let  $\varsigma_i$  be the least common multiple of  $\{\varsigma_{i,1}, \dots, \varsigma_{i,s-1}\}$ . Let  $g_i^j = \varphi_i^j(T = T^{\varsigma_i/\varsigma_{i,j}})$ . We call the set  $\mathfrak{G}_R := \{(X_1 = T^{\varsigma_i}, X_2 = g_i^1(T), \dots, X_s = g_i^{s-1}(T)), i = 1, \dots, M\}$  a system of Puiseux parametrizations of  $R$ .

**Theorem 3.** *We have*

$$\lim_0(W(R)) = \mathfrak{G}_R(T = 0).$$

*Proof.* It follows directly from Theorem 2 and Definition 2.  $\square$

## 5 Puiseux parametrization in finite accuracy

In this section, we define the Puiseux parametrizations of a polynomial  $f \in \mathbb{C}\langle X \rangle[Y]$  in finite accuracy, see Definition 4.

For  $f \in \mathbb{C}\langle X \rangle[Y]$ , we define the approximation  $\tilde{f}$  of  $f$  for a given finite accuracy, see Definition 3. This approximation  $\tilde{f}$  of  $f$  is a polynomial in  $\mathbb{C}[X, Y]$ . In Section 7, we prove that in order to compute a Puiseux parametrizations of  $f$  of a given accuracy, it suffices to compute a Puiseux parametrization of  $\tilde{f}$  of some finite accuracy.

In this section, we review and adapt the classical Newton-Puiseux algorithm to compute Puiseux parametrizations of a polynomial  $f \in \mathbb{C}[X, Y]$  of a given accuracy. Since we do not need to compute the singular part of Puiseux parametrizations, the usual requirement  $\text{discrim}(f, Y) \neq 0$  is dropped.

**Definition 3.** Let  $f = \sum_{i=0}^{\infty} a_i X^i \in \mathbb{C}[[X]]$ . For any  $\tau \in \mathbb{N}$ , let  $f^{(\tau)} := \sum_{i=0}^{\tau} a_i X^i$ . We call  $f^{(\tau)}$  the polynomial part of  $f$  of accuracy  $\tau + 1$ . Now, let  $f = \sum_{i=0}^d a_i(X) Y^i \in \mathbb{C}\langle X \rangle[Y]$ . For any  $\tau \in \mathbb{N}$ , we call  $\tilde{f}^{(\tau)} := \sum_{i=0}^d a_i^{(\tau)} Y^i$  the approximation of  $f$  of accuracy  $\tau + 1$ .

**Definition 4.** Let  $f \in \mathbb{C}\langle X \rangle[Y]$ ,  $\deg(f, Y) > 0$ . Let  $\sigma, \tau \in \mathbb{N}_{>0}$  and  $g(T) = \sum_{k=0}^{\tau-1} b_k T^k$ . The pair  $(T^\sigma, g(T))$  is called a Puiseux parametrization of  $f$  of accuracy  $\tau$  if there exists an irreducible Puiseux parametrization  $(T^\varsigma, \varphi(T))$  of  $f$  such that

- (i)  $\sigma$  divides  $\varsigma$ .
- (ii)  $\gcd(\sigma, b_0, \dots, b_{\tau-1}) = 1$ .
- (iii)  $g(T^{\varsigma/\sigma})$  is the polynomial part of  $\varphi(T)$  of accuracy  $(\varsigma/\sigma)(\tau - 1) + 1$ .

Note that if  $\sigma = \varsigma$ , then  $g(T)$  is simply the polynomial part of  $\varphi(T)$  of accuracy  $\tau$ .

We borrow the following notion from [14] in order to state an algorithm for computing Puiseux parametrizations.

**Definition 5.** A  $\mathbb{C}$ -term<sup>3</sup> is defined as a triple  $t = (q, p, \beta)$ , where  $q$  and  $p$  are coprime integers,  $q > 0$  and  $\beta \in \mathbb{C}$  is non-zero. A  $\mathbb{C}$ -expansion is a sequence  $\pi = (t_1, t_2, \dots)$  of  $\mathbb{C}$ -terms, where  $t_i = (q_i, p_i, \beta_i)$ . We say that  $\pi$  is finite if there are only finitely many elements in  $\pi$ .

**Definition 6.** Let  $\pi = (t_1, \dots, t_N)$  be a finite  $\mathbb{C}$ -expansion. We define a pair  $(T^\sigma, g(T))$  of polynomials in  $\mathbb{C}[T]$  in the following manner:

- if  $N = 1$ , set  $\sigma = 1$ ,  $g(T) = 0$  and  $\delta_N = 0$ ,
- otherwise, let  $a := \prod_{i=1}^N q_i$ ,  $c_i := \sum_{j=1}^i \left( p_j \prod_{k=j+1}^N q_k \right)$  ( $1 \leq i \leq N$ ), and  $\delta_i := c_i / \gcd(a, c_1, \dots, c_N)$  ( $1 \leq i \leq N$ ). Set  $\sigma := a / \gcd(a, c_1, \dots, c_N)$  and  $g(T) := \sum_{i=1}^N \beta_i T^{\delta_i}$ .

---

<sup>3</sup>It is a simplified version of Duval's definition.

We call the pair  $(T^\sigma, g(T))$  the corresponding Puiseux parametrization of  $\pi$  of accuracy  $\delta_N + 1$ . Denote by **ConstructParametrization** an algorithm to compute  $(T^\sigma, g(T))$  from  $\pi$ .

**Definition 7.** Let  $f \in \mathbb{C}\langle X \rangle[Y]$  and write  $f$  as  $f(X, Y) := \sum_{i=0}^d \left( \sum_{j=0}^{\infty} a_{i,j} X^j \right) Y^i$ . The Newton Polygon of  $f$  is defined as the lower part of the convex hull of the set of points  $(i, j)$  in the plane such that  $a_{i,j} \neq 0$ .

Let  $f \in \mathbb{C}\langle X \rangle[Y]$ . Next we present an algorithm, called **NewtonPolygon** to compute the segments in the Newton Polygon of  $f$ . This algorithm is from R.J. Walker's book [28].

**NewtonPolygon**( $f, I$ )

**Input:** A polynomial  $f \in \mathbb{C}\langle X \rangle[Y]$ ; a controlling flag  $I$ , whose value is 1 or 2.

**Output:** The Newton Polygon of  $f$ . If  $I = 1$ , only segments with non-positive slopes are computed. If  $I = 2$ , only segments with negative slopes are computed.

**Description:**

- Write  $f$  as  $f = \sum_{i=0}^d b_i(X) Y^i$ , where  $b_i(X) = \sum_{j=0}^{\infty} a_{i,j} X^j$ .
- For  $0 \leq i \leq d$ , define  $\delta_i := \text{ord}(b_i)$ .
- For  $0 \leq i \leq d$ , we plot the points  $P_i$  with coordinates  $(i, \delta_i)$ ; we omit  $P_i$  if  $\delta_i = \infty$ .
- We join  $P_0$  to  $P_d$  with a convex polygonal arc each of whose vertices is a  $P_i$  and such that no  $P_i$  lies below the arc.
- If  $I = 1$ , output all segments with non-positive slopes in the polygon; if  $I = 2$ , output all segments with negative slopes in the polygon.

Next we present the specification of several other sub-algorithms which are necessary to present Algorithm 2 for computing Puiseux parametrization of some finite accuracy as defined in Definition 4.

**NewPolynomial**( $f, t, \ell$ )

**Input:**  $f \in \mathbb{C}[X, Y]$ ; a  $\mathbb{C}$ -term  $t = (q, p, \beta)$ ;  $\ell \in \mathbb{N}$ .

**Output:** A polynomial  $X^{-\ell} f(X^q, X^p(\beta + Y)) \in \mathbb{C}[X, Y]$ .

**SegmentPoly**( $f, \Delta$ )

**Input:**  $f \in \mathbb{C}[X, Y]$ ;  $\Delta$  is a segment of the Newton Polygon of  $f$ .

**Output:** A quadruple  $(q, p, \ell, \phi)$  such that the following holds

- $q, p, \ell \in \mathbb{N}$ ;  $\phi \in \mathbb{C}[Z]$ ;  $q$  and  $p$  are coprime,  $q > 0$ .
- For any  $(i, j) \in \Delta$ , we have  $qj + pi = \ell$ .
- Let  $i_0 := \min(\{i \mid (i, j) \in \Delta\})$ , we have  $\phi = \sum_{(i,j) \in \Delta} a_{i,j} Z^{(i-i_0)/q}$ .

**Theorem 4.** Algorithm 2 terminates and is correct.

*Proof.* It directly follows from the proof of Newton-Puiseux algorithm in Walker's book [28], the relation between  $\mathbb{C}$ -expansion and Puiseux parametrization discussed in Duval's paper [14], and Definitions 6 and 4.  $\square$

---

**Algorithm 1:** NonzeroTerm( $f, I$ )

---

**Input:**  $f \in \mathbb{C}[X, Y]$ ;  $I = 1$  or  $2$   
**Output:** A finite set of pairs  $(t, \ell)$ , where  $t$  is a  $\mathbb{C}$ -term, and  $\ell \in \mathbb{N}$ .

```
1 begin
2    $S := \emptyset$ ;
3   for each  $\Delta \in \text{NewtonPolygon}(f, I)$  do
4      $(q, p, \ell, \phi) := \text{SegmentPoly}(f, \Delta)$ ;
5     for each root  $\xi$  of  $\phi$  in  $\mathbb{C}$  do
6       for each root  $\beta$  of  $U^q - \xi$  in  $\mathbb{C}$  do
7          $t := (q, p, \beta)$ ;
8          $S := S \cup \{(t, \ell)\}$ 
9   end
```

---

---

**Algorithm 2:** NewtonPuisseux

---

**Input:**  $f \in \mathbb{C}[X, Y]$ ; a given accuracy  $\tau > 0 \in \mathbb{N}$ .  
**Output:** All the Puiseux parametrizations of  $f$  of accuracy  $\tau$ .

```
1 begin
2    $\pi := ()$ ;  $S := \{(\pi, f)\}$ ;
3   while  $S \neq \emptyset$  do
4     choose  $(\pi^*, f^*) \in S$ ;  $S := S \setminus \{(\pi^*, f^*)\}$ ;
5     if  $\pi^* = ()$  then  $I := 1$  else  $I := 2$ ;
6      $(T^\sigma, g(T)) := \text{ConstructParametrization}(\pi^*)$ ;
7     if  $\deg(g(T), T) + 1 < \tau$  then
8        $C := \text{NonzeroTerm}(f^*, I)$ ;
9       if  $C = \emptyset$  then
10        output  $(T^\sigma, g(T))$  // a finite Puiseux parametrization is
            found
11      else
12        for each  $(t = (p, q, \beta), \ell) \in C$  do
13           $\pi^{**} := \pi^* \cup (t)$ ;
14           $f^{**} := \text{NewPolynomial}(f^*, t, \ell)$ ;
15           $S := S \cup \{(\pi^{**}, f^{**})\}$ 
16    output  $(T^\sigma, g(T))$ 
17 end
```

---

## 6 Computing in finite accuracy

Let  $f \in \mathbb{C}\langle X \rangle[Y]$ . In this section, we consider the following problems.

- (a) Is it possible to use an approximation of  $f$  of some finite accuracy  $m$  in order to compute a Puiseux parametrization of  $f$  of some finite accuracy  $\tau$ ?
- (b) If yes, how to deduce  $m$  from  $f$  and  $\tau$ ?
- (c) Provide a bound on  $m$ .

Theorem 5 provides the answers to (a) and (b) while Lemma 15 answers (c).

**Lemma 12** ([15]). *Let  $\underline{X} = X_1, \dots, X_s$  and  $\underline{Y} = Y_1, \dots, Y_m$ . For  $g_1, \dots, g_s \in \mathbb{C}[[\underline{Y}]]$ , with  $\text{ord}(g_i) \geq 1$ , there is a  $\mathbb{C}$ -algebra homomorphism (called the substitution homomorphism)*

$$\begin{aligned} \Phi_g : \quad \mathbb{C}[[\underline{X}]] &\longrightarrow \mathbb{C}[[\underline{Y}]] \\ f &\longmapsto f(g_1(\underline{Y}), \dots, g_s(\underline{Y})). \end{aligned}$$

Moreover, if  $g_1, \dots, g_s$  are convergent power series, then we have  $\Phi_g(\mathbb{C}\langle \underline{X} \rangle) \subseteq \mathbb{C}\langle \underline{Y} \rangle$  holds.

**Definition 8** ([15]). *Let  $f = \sum a_{\mu\nu} X^\mu Y^\nu \in \mathbb{C}[[X, Y]]$ . The carrier of  $f$  is defined as*

$$\text{carr}(f) = \{(\mu, \nu) \in \mathbb{N}^2 \mid a_{\mu\nu} \neq 0\}.$$

**Lemma 13.** *Let  $f \in \mathbb{C}\langle X \rangle[Y]$ . Let  $d := \deg(f, Y) > 0$ . Let  $q \in \mathbb{N}_{>0}$ ,  $p, \ell \in \mathbb{N}$  and assume that  $q$  and  $p$  are coprime. Let  $\beta \neq 0 \in \mathbb{C}$ . Assume that  $q, p, \ell$  define a line  $L : qj + pi = \ell$  in  $(i, j)$  plane such that*

- (a) *There are at least two points  $(j_1, i_1) \in \text{carr}(f)$  and  $(j_2, i_2) \in \text{carr}(f)$  on  $L$  with  $i_1 \neq i_2$ .*
- (b) *For any  $(j, i) \in \text{carr}(f)$ , we have  $qj + pi \geq \ell$ .*

*Let  $f_1 := X_1^{-\ell} f(X_1^q, X_1^p(\beta + Y_1))$ . Then, we have the following results*

- (i) *We have  $f_1 \in \mathbb{C}\langle X_1 \rangle[Y_1]$ .*
- (ii) *For any given  $m_1 \in \mathbb{N}$ , there exists a finite number  $m \in \mathbb{N}$  such that the approximation of  $f_1$  of accuracy  $m_1$  can be computed from the approximation of  $f$  of accuracy  $m$ .*
- (iii) *Moreover, it suffices to take  $m = \lfloor \frac{m_1 + \ell}{q} \rfloor$ .*

*Proof.* Since  $q > 0$  holds, we know that  $\text{ord}(X_1^q) = q > 0$  holds. We also have  $f(X_1^q, X_1^p(\beta + Y_1)) \in \mathbb{C}\langle X_1 \rangle[Y_1]$ . Let  $f(X, Y) := \sum_{i=0}^d \left( \sum_{j=0}^{\infty} a_{i,j} X^j \right) Y^i$ . Then we have  $f_1(X_1, Y_1) = \sum_{i=0}^d \left( \sum_{j=0}^{\infty} a_{i,j} X_1^{(qj+pi-\ell)} \right) (\beta + Y_1)^i$ . Since for any  $(j, i) \in \text{carr}(f)$ , we have  $qj + pi \geq \ell$ , the power of  $X_1$  cannot be negative. By Lemma 12, we have  $f_1 \in \mathbb{C}\langle X_1 \rangle[Y_1]$ . That is (i) holds.

We prove (ii). We have

$$\begin{aligned} &f_1(X_1, Y_1) \bmod \langle X_1^{m_1} \rangle \\ &= \sum_{i=0}^d \left( \sum_{qj+pi-\ell < m_1} a_{i,j} X_1^{(qj+pi-\ell)} \right) (\beta + Y_1)^i. \end{aligned}$$

Since  $q \in \mathbb{N}_{>0}$  and  $m_1, \ell$  and  $i$  are all finite, we know that  $j$  has to be finite. In other words, there exists a finite  $m$  such that the approximation of  $f_1$  of accuracy  $m_1$  can be computed from the approximation of  $f$  of accuracy  $m$ . That is, (ii) holds.

Since the first  $m_1$  terms of  $f_1$  depends on the  $j$ -th terms of  $f$ , which satisfies the constraint  $qj + pi - \ell < m_1$ , we have  $j < \frac{(m_1 + \ell) - pi}{q} \leq \frac{(m_1 + \ell)}{q}$ . Let  $m'$  be the maximum of these  $j$ 's. Now we have  $m' - 1 < \frac{(m_1 + \ell)}{q}$ . Since  $m'$  is an integer, we have  $m' \leq \lfloor \frac{(m_1 + \ell)}{q} \rfloor$  holds. Let  $m = \lfloor \frac{(m_1 + \ell)}{q} \rfloor$ . Next we show that  $m_1 \geq 1$  implies that  $m \geq 1$  holds. If there is

at least one point  $(i, j) \in L$  such that  $j \geq 1$ , then we have  $\ell \geq q$ , which implies  $m \geq 1$ . If the  $j$ -coordinates of all points on  $L$  is 0, then  $q = 1$  and  $\ell = 0$ , which implies also  $m \geq 1$ . Thus (iii) is proved.  $\square$

**Remark 1.** We use the same notations as in the previous Lemma. In particular, let  $f(X, Y) := \sum_{i=0}^d \left( \sum_{j=0}^{\infty} a_{i,j} X^j \right) Y^i$  and  $f_1 := X_1^{-\ell} f(X_1^q, X_1^p(\beta + Y_1))$ . For a fixed term  $a_{i,j} X^j Y^i$  of  $f$ , it appears in  $f_1$  as

$$a_{i,j} X_1^{qj+pi-\ell} (\beta + Y_1)^i = \sum_{k=0}^i \binom{i}{k} \beta^{i-k} a_{i,j} X_1^{qj+pi-\ell} Y_1^k.$$

For two fixed terms  $a_{i,j_1} X^{j_1} Y^i$  and  $a_{i,j_2} X^{j_2} Y^i$  of  $f$  with  $j_1 < j_2$ , since  $qj_1 + pi - \ell < qj_2 + pi - \ell$ , we know that for any fixed  $k$ ,  $a_{i,j_2} X^{j_2} Y^i$  always contributes strictly higher order of powers of  $X_1$  than  $a_{i,j_1} X^{j_1} Y^i$  in  $f_1$ .

**Remark 2.** Let  $f(X, Y) := \sum_{i=0}^d \left( \sum_{j=0}^{\infty} a_{i,j} X^j \right) Y^i$ . For  $0 \leq i \leq d$ , let  $a_{i,j^*}$  be the first nonzero coefficient among  $\{a_{i,j} | 0 \leq j < \infty\}$ . We observe that the Newton polygon of  $f$  is completely determined by  $a_{i,j^*}$ ,  $0 \leq i \leq d$ .

**Theorem 5.** Let  $f \in \mathbb{C}\langle X \rangle[Y]$ . Let  $\tau \in \mathbb{N}_{>0}$ . Let  $\sigma \in \mathbb{N}_{>0}$  and  $g(T) = \sum_{k=0}^{\tau-1} b_k T^k$ . Assume that  $(T^\sigma, g(T))$  is a Puiseux parametrization of  $f$  of accuracy  $\tau$ . Then one can compute a finite number  $m \in \mathbb{N}$  such that  $(T^\sigma, g(T))$  is a Puiseux parametrization of accuracy  $\tau$  of the approximation of  $f$  of accuracy  $m$ . We denote by **AccuracyEstimate** an algorithm to compute such  $m$  from  $f$  and  $\tau$ .

*Proof.* Let  $f_0 := f$ ,  $X_0 := X$  and  $Y_0 := Y$ . For  $i = 1, 2, \dots$ , Newton-Puiseux's algorithm computes numbers  $q_i, p_i, \ell_i, \beta_i$  and the transformation

$$f_i := X_i^{-\ell_i} f_{i-1}(X_i^{q_i}, X_i^{p_i}(\beta_i + Y_i))$$

such that the assumption of Lemma 13 is satisfied.

By Lemma 13, we know that for any  $i$ , a given number of terms of the coefficients of  $f_i$  in  $Y_i$  can be computed from a finite number of terms of the coefficients of  $f_{i-1}$  in  $Y_{i-1}$ . Thus for any  $i$ , a given number of terms of the coefficients of  $f_i$  in  $Y_i$  can be computed from a finite number of terms of the coefficients of  $f$  in  $Y$ .

On the other hand, the construction of Newton-Puiseux's algorithm and Remark 2 tell us that there exists a finite  $M$ , such that  $\sigma$  and all the terms of  $g(T)$  can be computed from a finite number of terms of the coefficients of  $f_i$  in  $Y_i$ ,  $i = 1, \dots, M$ .

Thus we conclude that there exists a finite number  $m \in \mathbb{N}$  such that  $(T^\sigma, g(T))$  is a Puiseux parametrization of accuracy  $\tau$  of the approximation of  $f$  of accuracy  $m$ .

Next we show that there is an algorithm to compute  $m$ . We initially set  $m' := \tau$ . Let  $f_0 := \sum_{i=0}^d \left( \sum_{j=0}^{m'} a_{i,j} X^j \right) Y^i$ . That is,  $f_0$  is the approximation of  $f$  of accuracy  $m' + 1$ . We run Newton-Puiseux's algorithm to check whether the terms  $a_{k,m'} X^{m'} Y^k$ ,  $0 \leq k \leq d$ , make any contributions in constructing the Newton Polygons of all  $f_i$ . If at least one of them make contributions, we increase the value of  $m'$  and restart the Newton-Puiseux's algorithm until none of the terms  $a_{k,m'} X^{m'} Y^k$ ,  $0 \leq k \leq d$ , makes any contributions in constructing Newton Polygons of all  $f_i$ . By Remark 1, we can set  $m := m'$ .  $\square$

**Lemma 14.** Let  $d, \tau \in \mathbb{N}_{>0}$ . Let  $a_{i,j}$ ,  $0 \leq i \leq d$ ,  $0 \leq j < \tau$ , and  $b_k$ ,  $0 \leq k < \tau$  be symbols. Write  $\mathbf{a} = (a_{0,0}, \dots, a_{0,\tau-1}, \dots, a_{d,0}, \dots, a_{d,\tau-1})$  and  $\mathbf{b} = (b_0, \dots, b_{\tau-1})$ . Let  $f(\mathbf{a}, X, Y) = \sum_{i=0}^d \left( \sum_{j=0}^{\tau-1} a_{i,j} X^j \right) Y^i \in \mathbb{C}[\mathbf{a}][X, Y]$ . Let  $g(\mathbf{b}, X) = \sum_{k=0}^{\tau-1} b_k X^k \in \mathbb{C}[\mathbf{b}][X]$ . Let  $p := f(\mathbf{a}, X, Y = g(\mathbf{b}, X))$ . Let  $F_k := \text{coeff}(p, X^k)$ ,  $0 \leq k < \tau - 1$ , and  $F := \{F_0, \dots, F_{\tau-1}\}$ . Then under the order  $\mathbf{a} < \mathbf{b}$  and  $b_0 < b_1 < \dots < b_{\tau-1}$ ,  $F$  forms a zero-dimensional regular chain in  $\mathbb{C}(\mathbf{a})[\mathbf{b}]$  with main variables  $(b_0, b_1, \dots, b_{\tau-1})$  and main degrees  $(d, 1, \dots, 1)$ . In addition, we have

- $F_0 = \sum_{i=0}^d a_{i,0} b_0^i$  and
- $\text{init}(F_1) = \dots = \text{init}(F_{\tau-1}) = \sum_{i=1}^d i \cdot a_{i,0} b_0^{i-1}$ .

*Proof.* Write  $p = \sum_{i=0}^d \left( \sum_{j=0}^{\tau-1} a_{i,j} X^j \right) \left( \sum_{k=0}^{\tau-1} b_k X^k \right)^i$  as a univariate polynomial in  $X$ . Observe that  $F_0 = \sum_{i=0}^d a_{i,0} b_0^i$ . Therefore  $F_0$  is irreducible in  $\mathbb{C}(\mathbf{a})[\mathbf{b}]$ . Moreover, we have  $\text{mvar}(F_0) = b_0$  and  $\text{mdeg}(F_0) = d$ .

Since  $d > 0$ , we know that  $a_{1,0} \left( \sum_{k=0}^{\tau-1} b_k X^k \right)$  appears in  $p$ . Thus, for  $0 \leq k < \tau$ ,  $b_k$  appears in  $F_k$ . Moreover, for any  $k \geq 1$  and  $i < k$ ,  $b_k$  can not appear in  $F_i$  since  $b_k$  and  $X^k$  are always raised to the same power. For the same reason, for any  $i > 1$ ,  $b_k^i$  cannot appear in  $F_k$ , for  $1 \leq k < \tau$ . Thus  $\{F_0, \dots, F_{\tau-1}\}$  is a triangular set with main variables  $(b_0, b_1, \dots, b_{\tau-1})$  and main degrees  $(d, 1, \dots, 1)$ .

Moreover, we have  $\text{init}(F_1) = \dots = \text{init}(F_{\tau-1}) = \sum_{i=1}^d i \cdot a_{i,0} b_0^{i-1}$ , which is coprime with  $F_0$ . Thus  $F = \{F_0, \dots, F_{\tau-1}\}$  is a regular chain.  $\square$

**Lemma 15.** Let  $f = \sum_{i=0}^d \left( \sum_{j=0}^{\infty} a_{i,j} X^j \right) Y^i \in \mathbb{C}[[X]][Y]$ . Assume that  $\deg(f, Y) > 0$  and  $f$  is general in  $Y$ . Let  $\varphi(X) = \sum_{k=0}^{\infty} b_k X^k \in \mathbb{C}[[X]]$  such that  $f(X, \varphi(X)) = 0$  holds. Let  $\tau > 0 \in \mathbb{N}$ . Then “generically”,  $b_i$ ,  $0 \leq i < \tau$ , can be completely determined by  $\{a_{i,j} \mid 0 \leq i \leq d, 0 \leq j < \tau\}$ .

*Proof.* By  $f(X, Y) = 0$ , we know that  $f(X, Y) = 0 \bmod \langle X^\tau \rangle$ . Therefore, we have

$$\sum_{i=0}^d \left( \sum_{j < \tau} a_{i,j} X^j \right) \left( \sum_{k < \tau} b_k X^k \right)^i = 0 \bmod \langle X^\tau \rangle.$$

Let  $p = \sum_{i=0}^d \left( \sum_{j < \tau} a_{i,j} X^j \right) \left( \sum_{k < \tau} b_k X^k \right)^i$ . Let  $F_i := \{\text{coeff}(p, X^i), 0 \leq i < \tau\}$ , and  $F := \{F_0, \dots, F_{\tau-1}\}$ . Since  $f$  is general in  $Y$  and  $f(X, \varphi(X)) = 0$ , there exists  $i^* > 0$  such that  $a_{i^*,0} \neq 0$ . By Lemma 14, we have  $F_0 = \sum_{i=0}^d a_{i,0} b_0^i$ . Thus  $b_0$  can be completely determined by  $a_{i,0}$ ,  $0 \leq i \leq d$ . In order to completely determine  $b_1, \dots, b_{\tau-1}$ , it is enough to guarantee  $\text{res}(F_0, F_i, b_0) \neq 0$  holds. Therefore the values of  $b_k$ ,  $0 \leq k < \tau$  can be completely determined from almost all the values of  $a_{i,j}$ ,  $0 \leq i \leq d$ ,  $0 \leq j < \tau$ .  $\square$

## 7 Accuracy estimates

Let  $R := \{r_1(X_1, X_2), \dots, r_{s-1}(X_1, \dots, X_s)\} \subset \mathbb{C}[X_1 < \dots < X_s]$  be a strongly normalized regular chain. In this section, we show that to compute the limit points of  $W(R)$ , it suffices to compute the Puiseux parametrizations of  $R$  of some accuracy. Moreover, we provide accuracy estimates in Theorem 6.



**Lemma 16.** Let  $f = a_d(X)Y^d + \dots + a_0(X) \in \mathbb{C}\langle X \rangle[Y]$ , where  $d > 0$  and  $a_d(X) \neq 0$ . For  $0 \leq i \leq d$ , let  $\delta_i := \text{ord}(a_i)$ . Let  $k := \min(\delta_0, \dots, \delta_d)$ . Let  $\tilde{f} := f/X^k$ . Then we have  $\tilde{f} \in \mathbb{C}\langle X \rangle[Y]$  and  $\tilde{f}$  is general in  $Y$ . This process of producing  $\tilde{f}$  from  $f$  is called “making  $f$  general” and denote by **MakeGeneral** an operation which produces  $\tilde{f}$  from  $f$ .

*Proof.* Since  $k = \min(\delta_0, \dots, \delta_d)$ , there exists  $i$ ,  $1 \leq i \leq d$ , such that  $k = \delta_i$ . Moreover, for all  $1 \leq j \leq d$ , we have  $\delta_j \geq k$ . Thus for every such  $i$ , we have  $\text{ord}(a_i(X)/X^k) = 0$  and  $a_j(X)/X^k \in \mathbb{C}\langle X \rangle$ ,  $0 \leq j \leq d$ . This shows that  $\tilde{f} \in \mathbb{C}\langle X \rangle[Y]$  and  $\tilde{f}$  is general in  $Y$ .  $\square$

The following lemma shows that computing limit points reduces to making a polynomial  $f$  general.

**Lemma 17.** Let  $f \in \mathbb{C}\langle X \rangle[Y]$ , where  $\deg(f, Y) > 0$ . Assume that  $f$  is general in  $Y$ . Let  $\rho > 0$  be small enough such that  $f$  converges in  $|X| < \rho$ . Let  $V_\rho(f) := \{(x, y) \in \mathbb{C}^2 \mid 0 < |x| < \rho, f(x, y) = 0\}$ . Then we have  $\lim_0(V_\rho(f)) = \{(0, y) \in \mathbb{C}^2 \mid f(0, y) = 0\}$ .

*Proof.* Let  $(X = T^{\varsigma_i}, Y = \varphi_i(T))$ ,  $1 \leq i \leq c \leq d$ , be the Puiseux parametrizations of  $f$ . By Lemma 9 and Theorem 3, we have  $\lim_0(V_\rho(f)) = \cup_{i=1}^c \{(0, y) \in \mathbb{C}^2 \mid y = \varphi_i(0)\}$ . Let  $(X = T^{\sigma_i}, g_i(T))$ ,  $i = 1, \dots, c$ , be the corresponding Puiseux parametrizations of  $f$  of accuracy 1. By Theorem 5, there exists an approximation  $\tilde{f}$  of  $f$  of some finite accuracy such that  $(X = T^{\sigma_i}, g_i(T))$ ,  $i = 1, \dots, c$ , are also Puiseux parametrizations of  $\tilde{f}$  of accuracy 1. Thus, we have  $\varphi_i(0) = g_i(0)$ ,  $i = 1, \dots, c$ . Since  $\tilde{f}$  is also general in  $Y$ , by Theorem 2.3 of Walker [28], we have  $\cup_{i=1}^c \{(0, y) \in \mathbb{C}^2 \mid y = g_i(0)\} = \{(0, y) \in \mathbb{C}^2 \mid \tilde{f}(0, y) = 0\}$ . Since  $\tilde{f}(0, y) = f(0, y)$ , the Lemma holds.  $\square$

**Lemma 18.** Let  $a(X_1, \dots, X_s) \in \mathbb{C}[X_1, \dots, X_s]$ . Let  $g_i = \sum_{j=0}^\infty c_{ij}T^j \in \mathbb{C}\langle T \rangle$ . We write  $a(g_1, \dots, g_s)$  as  $\sum_{k=0}^\infty b_k T^k$ . To compute a given  $b_k$ , one only needs the set of coefficients  $\{c_{i,j} \mid 1 \leq i \leq s, 0 \leq j \leq k\}$ .

*Proof.* We observe that any  $c_{i,j}$ , where  $j > k$ , does not make any contribution to  $b_k$ .  $\square$

**Lemma 19.** Let  $f = a_d(X)Y^d + \dots + a_0(X) \in \mathbb{C}\langle X \rangle[Y]$ , where  $d > 0$ , and  $a_d(X) \neq 0$ . Let  $\delta := \text{ord}(a_d(X))$ . Then “generically”, a Puiseux parametrization of  $f$  of accuracy  $\tau$  can be computed from an approximation of  $f$  of accuracy  $\tau + \delta$ .

*Proof.* Let  $\tilde{f} := \text{MakeGeneral}(f)$ . Observe that  $f$  and  $\tilde{f}$  have the same system of Puiseux parametrizations. Then the conclusion follows from Lemma 16 and Lemma 15.  $\square$

**Theorem 6.** Let  $R := \{r_1(X_1, X_2), \dots, r_{s-1}(X_1, \dots, X_s)\} \subset \mathbb{C}[X_1 < \dots < X_s]$  be a strongly normalized regular chain. For  $1 \leq i \leq s-1$ , let  $h_i := \text{init}(r_i)$ ,  $d_i := \deg(r_i, X_{i+1})$  and  $\delta_i := \text{ord}(h_i)$ . We define  $f_i$ ,  $2 \leq i \leq s-1$ , and  $\varsigma_j$ ,  $T_j$ ,  $\varphi_j(T_j)$ ,  $1 \leq j \leq s-2$ , as follows

- Let  $(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1))$  be a Puiseux parametrization of  $r_1(X_1, X_2)$ .
- Let  $f_i := r_i(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \dots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1})$ .
- Let  $(T_{i-1} = T_i^{\varsigma_i}, X_{i+1} = \varphi_i(T_i))$  be a Puiseux parametrization of  $f_i$ .

Then we have the following results:

- Let  $T_0 := X_1$ , for  $0 \leq i \leq s-2$ , define  $g_i(T_{s-2}) := T_{s-2}^{\prod_{k=i+1}^{s-2} \varsigma_k}$ , then we have  $T_i = g_i(T_{s-2})$ .
- We have  $f_{s-1} \in \mathbb{C}\langle T_{s-2} \rangle[X_s]$ .

- (iii) There exist numbers  $\tau_1, \dots, \tau_{s-2} \in \mathbb{N}$  such that in order to make  $f_{s-1}$  general in  $X_s$ , it suffices to compute the polynomial parts of  $\varphi_i$  of accuracy  $\tau_i$ ,  $1 \leq i \leq s-2$ . Moreover, if we write the algorithm **AccuracyEstimate** for short as  $\theta$ , the accuracies  $\tau_i$  can be computed in the following manner
- let  $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k) \delta_{s-1} + 1$
  - let  $\tau_{i-1} := \max(\theta(f_i, \tau_i), (\prod_{k=1}^{i-1} \varsigma_k) \delta_{s-1} + 1)$ , for  $s-2 \geq i \geq 2$ .
- (iv) Generically, for  $1 \leq i \leq s-3$ , we can choose  $\tau_i = (\prod_{k=1}^{s-2} \varsigma_k)(\sum_{k=2}^{s-1} \delta_i) + 1$ .
- (v) The indices  $\varsigma_k$  can be replaced with  $d_k$ ,  $k = 1, \dots, s-2$ .

*Proof.* We prove (i) by induction. Clearly (i) holds for  $i = s-2$ . Suppose it holds for  $i$ . Then we have

$$T_{i-1} = T_i^{\varsigma_i} = \left( T_{s-2}^{\prod_{k=i+1}^{s-2} \varsigma_k} \right)^{\varsigma_i} = \left( T_{s-2}^{\prod_{k=i}^{s-2} \varsigma_k} \right)$$

Therefore (i) holds also for  $i-1$ . So (i) holds for all  $0 \leq i \leq s-2$ .

Note that

$$\begin{aligned} f_{s-1} &= r_{s-1}(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \dots, \\ &\quad X_{s-2} = \varphi_{s-3}(T_{s-3}), X_{s-1} = \varphi_{s-2}(T_{s-2}), X_s) \\ &= r_{s-1}(X_1 = g_0(T_{s-2}), X_2 = \varphi_1(g_1(T_{s-2})), \dots, \\ &\quad X_{s-2} = \varphi_{s-3}(g_{s-3}(T_{s-2})), \\ &\quad X_{s-1} = \varphi_{s-2}(g_{s-2}(T_{s-2})), X_s) \end{aligned} \quad (5)$$

Since  $\text{ord}(g_i(T_{s-2})) > 0$  for all  $0 \leq i \leq s-2$ , by Lemma 12, (ii) holds.

Note that  $g_0(T_{s-2}) = T_{s-2}^{\prod_{k=1}^{s-2} \varsigma_k}$ . Since  $\text{ord}(h_{s-1}(X_1)) = \delta_{s-1}$ , we have

$$\text{ord}(h_{s-1}(X_1 = g_0(T_{s-2}))) = \left( \prod_{k=1}^{s-2} \varsigma_k \right) \delta_{s-1}.$$

Let  $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k) \delta_{s-1} + 1$ . By Lemma 16, to make  $f_{s-1}$  general in  $X_s$ , it suffices to compute the polynomial parts of the coefficients of  $f_{s-1}$  of accuracy  $\tau_{s-2}$ .

By Lemma 18, and Equation (5), we need to compute the polynomial parts of  $\varphi_i(g_i(T_{s-2}))$ ,  $1 \leq i \leq s-2$ , of accuracy  $\tau_{s-2}$ . Since  $\text{ord}(g_i(T_{s-2})) = \prod_{k=i+1}^{s-2} \varsigma_k$ , to achieve this accuracy, it's enough to compute the polynomial parts of  $\varphi_i$  of accuracy  $(\prod_{k=1}^i \varsigma_k) \delta_{s-1} + 1$ , for  $1 \leq i \leq s-2$ .

On the other hand, since  $f_i = r_i(X_1 = T_1^{\varsigma_1}, X_2 = \varphi_1(T_1), \dots, X_i = \varphi_{i-1}(T_{i-1}), X_{i+1})$  and  $(T_{i-1} = T_i^{\varsigma_i}, X_{i+1} = \varphi_i(T_i))$  is a Puiseux parametrization of  $f_i$ , by Theorem 5 and Lemma 18, to compute the polynomial part of  $\varphi_i$  of accuracy  $\tau_i$ , we need the polynomial part of  $\varphi_{i-1}$  of accuracy  $\theta(f_i, \tau_i)$ .

Thus, take  $\tau_{s-2} := (\prod_{k=1}^{s-2} \varsigma_k) \delta_{s-1} + 1$  and  $\varphi_{i-1} = \max(\theta(f_i, \tau_i), (\prod_{k=1}^{i-1} \varsigma_k) \delta_{s-1} + 1)$  for  $2 \leq i \leq s-2$  will guarantee  $f_{s-1}$  can be made general in  $X_s$ . So (iii) holds. By Lemma 19, generically we can choose  $\theta(f_i, \tau_i) = \tau_i + (\prod_{k=1}^{i-1} \varsigma_k) \delta_i$ ,  $2 \leq i \leq s-2$ . Therefore (iv) holds. Since we have  $\varsigma_k \leq d_k$ ,  $1 \leq k \leq s-2$ , (v) holds.  $\square$

## 8 Algorithm

In this section, we provide a complete algorithm for computing the non-trivial limit points of the quasi-component of a one-dimensional strongly normalized regular chain based on the results of the previous sections.

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**Algorithm 3:** LimitPointsAtZero

---

**Input:**  $R := \{r_1(X_1, X_2), \dots, r_{s-1}(X_1, \dots, X_s)\} \subset \mathbb{C}[X_1 < \dots < X_s]$ ,  $s > 1$ , is a strongly normalized regular chain.

**Output:** The non-trivial limit points of  $W(R)$  whose  $X_1$ -coordinates are 0.

```
1 begin
2   let  $S := \{(T_0)\}$ ;
3   compute the accuracy estimates  $\tau_1, \dots, \tau_{s-2}$  by Theorem 6; let  $\tau_{s-1} = 1$ ;
4   for  $i$  from 1 to  $s - 1$  do
5      $S' := \emptyset$ ;
6     for  $\Phi \in S$  do
7        $f_i := r_i(X_1 = \Phi_1, \dots, X_i = \Phi_i, X_{i+1})$ ;
8       if  $i > 1$  then
9         let  $\delta := \text{ord}(f_i, T_{i-1})$ ; let  $f_i := f_i / T_{i-1}^\delta$ ;
10         $E := \text{NewtonPuisseux}(f_i, \tau_i)$ ;
11        for  $(T_{i-1} = \phi(T_i), X_{i+1} = \varphi(T_i)) \in E$  do
12           $S' := S' \cup \{\Phi(T_{i-1} = \phi(T_i)) \cup (\varphi(T_i))\}$ 
13       $S := S'$ 
14   if  $S = \emptyset$  then return  $\emptyset$  else
15     return eval( $S, T_{s-1} = 0$ )
16 end
```

---

---

**Algorithm 4:** LimitPoints

---

**Input:** A strongly normalized regular chain  
 $R := \{r_1(X_1, X_2), \dots, r_{s-1}(X_1, \dots, X_s)\} \subset \mathbb{C}[X_1 < \dots < X_s]$ ,  $s > 1$ .

**Output:** All the non-trivial limit points of  $W(R)$ .

```
1 begin
2   let  $h_R := \text{init}(R)$ ; let  $L$  be the set of roots of  $h_R$ ;
3    $S := \emptyset$ ;
4   for  $\alpha \in L$  do
5      $R_\alpha := R(X_1 = X_1 + \alpha)$ ;
6      $S_\alpha := \text{LimitPointsAtZero}(R_\alpha)$ ;
7     update  $S_\alpha$  by replacing the first coordinate of every point in it by  $\alpha$ ;
8      $S := S \cup S_\alpha$ 
9   return  $S$ 
10 end
```

---

**Remark 3.** Note that line 9 of Algorithm 3 computes Puiseux parametrizations of  $f_i$  of accuracy  $\tau_i$ . Thus  $(\phi(T_i), \varphi(T_i))$  at line 10 cannot have negative orders.

If the D5 principle is applied to Algorithms 3 and 4, the limit points of  $W(R)$  can be represented by a finite family of regular chains.

**Proposition 4.** Algorithm 4 is correct and terminates.

*Proof.* It follows from Theorem 3, Theorem 5, Theorem 6 and Lemma 17.  $\square$

## 9 Experimentation

We have implemented Algorithm 4 of Section 8, which computes the limit points of the quasi-component of a one-dimensional strongly normalized regular chain. The implementation is based on the library `RegularChains` and the command `algcures[puiseux]` of MAPLE. The code is available at <http://www.orcca.on.ca/~cchen/ACM13/LimitPoints.mpl>. This preliminary implementation relies on algebraic factorization, whereas, as suggested in [14], applying the D5 principle, in the spirit of triangular decomposition algorithms, for instance [8], would be sufficient when computations need to split into different cases. This would certainly improve performance greatly and this enhancement is work in progress.

As pointed out in the introduction, the computation of the limit points of the quasi-component of a regular chain can be applied to removing redundant components in a Kalkbrener triangular decomposition. In Table 1, we report on experimental results of this application.

The polynomial systems listed in this table are one-dimensional polynomial systems selected from the literature [6, 8]. For each system, we first call the `Triangularize` command of the library `RegularChains`, with the option “`normalized='strongly', 'radical'='yes'`”. For the input system, this process computes a Kalkbrener triangular decomposition  $\mathcal{R}$  where the regular chains are strongly normalized and their saturated ideals are radical. Next, for each one-dimensional regular chain  $R$  in the output, we compute the limit points  $\lim(W(R))$ , thus deducing a set of regular chains  $R_1, \dots, R_e$  such the union of their quasi-components equals the Zariski closure  $\overline{W(R)}$ . The algorithm `Difference` [6] is then called to test whether or not there exists a pair  $R, R'$  of regular chains of  $\mathcal{R}$  such that the inclusion  $\overline{W(R)} \subseteq \overline{W(R')}$  holds.

In Table 1, the column `T` and `#(T)` denote respectively the timings spent by `Triangularize` and the number of regular chains returned by this command; the column `d-1` and `d-0` denote respectively the number of 1-dimensional and 0-dimensional regular chains, whose sum is exactly `#(T)`; the column `R` and `#(R)` denote respectively the timings spent on removing redundant components in the output of `Triangularize` and the number of regular chains in the output irredundant decomposition. As we can see in the table, most of the decompositions are checked to be irredundant, which we could not do before this work by means of triangular decomposition algorithms. In addition, the three redundant 0-dimensional components in the Kalkbrener triangular decomposition of system f-744 are successfully removed. Therefore, we have verified experimentally the benefits provided by the algorithms presented in this paper.

Table 1: Removing redundant components.

Sys	T	#(T)	d-1	d-0	R	#(R)
f-744	14.360	4	1	3	432.567	1
Liu-Lorenz	0.412	3	3	0	216.125	3
MontesS3	0.072	2	2	0	0.064	2
Neural	0.296	5	5	0	1.660	5
Solotareff-4a	0.632	7	7	0	32.362	7
Vermeer	1.172	2	2	0	75.332	2
Wang-1991c	3.084	13	13	0	6.280	13

## 10 Concluding remarks

We conclude with a few remarks about special cases and a generalization of the algorithms presented in this paper.

**Reduction to strongly normalized chains.** Using the hypotheses of Lemma 3, we observe that one can reduce the computation of  $\lim(W(R))$  to that of  $\lim(W(N))$ . Indeed, under the assumption that  $\text{sat}(R)$  has dimension one, both  $\lim(W(R))$  and  $\lim(W(N))$  are finite. Once the set  $\lim(W(N))$  is computed, one can easily check which points in  $\lim(W(N))$  do not belong to  $W(R)$  and then deduce  $\lim(W(R))$ . This reduction to strongly normalized regular chains has the advantage that  $h_N$  is a univariate polynomial in  $\mathbb{C}[X_1]$ , which simplifies the presentation of the basic ideas of our algorithms, see Section 3. However, it has two drawbacks. First the coefficients of  $N$  are generally much larger than those of  $R$ . Secondly,  $\lim(W(N))$  may also be much larger than  $\lim(W(R))$ . A detailed presentation of a direct computation of  $\lim(W(R))$ , without reducing to  $\lim(W(N))$ , will be done in a future paper.

**Shape lemma case.** Here, by reference to the paper [2] (which deals with polynomial ideals of dimension zero) we assume that, for  $2 \leq i \leq e$ , the polynomial  $r_i$  involves only the variables  $X_1, X_2, X_i$  and that  $\deg(r_i, X_i) = 1$  holds. In this case, computing Puiseux series expansions is required only for the polynomial of  $R$  of lower rank, namely  $r_1$ . In this case, the algorithms presented in this paper are much simplified. However, for the specific purpose of solving polynomial systems via triangular decompositions, reducing to this Shape lemma case, via a random change of coordinates, has a negative impact on performance and software design, for many problems of practical interest. In contrast, the point of view of the work initiated in this paper is two-fold: first, deliver algorithms that do not require any genericity assumptions; second develop criteria that take advantage of specific properties of the input systems in order to speedup computations. Yet, in our implementation, several tricks are used to avoid unnecessary Puiseux series expansions, such as applying the theorem (see [15] p.113) on the continuity of the roots of a parametric polynomial.

**Handling the case where  $\text{sat}(R)$  has dimension greater than 1.** From now on,  $\text{sat}(R)$  has dimension  $s - e \geq 2$ . We use the notations of Lemma 5 and recall that each point of  $\lim(W(R' \cup r_e))$  is in particular a point of  $\lim(W(R'))$ . Since we know how to compute  $\lim(W(R'))$  when  $R'$  consists of a single polynomial, we assume, by induction that a triangular decomposition of  $\lim(W(R'))$  has been computed in the form  $W(R_1) \cup \dots \cup W(R_f)$  for regular chains  $R_1, \dots, R_f$ .

We observe that a point  $p \in \lim(W(R'))$  can be “extended” to a point of  $\lim(W(R' \cup r_e))$  in two ways. First, if  $p$  does not cancel the initial of  $r_e$  (which can be tested algorithmically), then, by applying the theorem (see again [15] p.113) on the continuity of the roots to  $r_e$ , we extend  $p$  with the  $X_s$ -roots of  $r_e$ , after specializing  $(X_1, \dots, X_{s-1})$  to  $p$ . From now

on, we assume that  $p$  cancels the initial  $h_e$  of  $r_e$ . In this case, we compute a truncated Puiseux parametrization about  $p$  using the regular chain  $R_i$  such that  $p \in W(R_i)$  holds. After substitution into the polynomial  $r_e$ , we apply Puiseux theorem and compute the limit points of  $W(R' \cup r_e)$  extending  $p$ , in the manner of the algorithms of Section 8.

There are new challenges, however, w.r.t. to the one-dimensional case. First, parametrizations may involve now more than one parameter. When this happens, one should use Jung-Abhyankar theorem [25] instead of the Puiseux theorem. The second difficulty is that  $\lim(W(R')) \cap V(h_e)$  may be infinite. This will not happen, however, if  $\text{sat}(R')$  has dimension at most 2 and  $h_e$  is regular w.r.t.  $\text{sat}(R')$ . This second assumption can be regarded as a genericity assumption. Thus the algorithms presented can easily be extended to dimension two, under that assumption, which can be tested algorithmically. Overcoming in higher dimensions this cardinality issue with  $\lim(W(R')) \cap V(h_e)$ , requires to understand which “configurations” are essentially the same. Since  $\lim(W(R))$ , as an algebraic set, can be described by finitely many regular chains, this is, indeed, possible and work in progress.

## References

- [1] M. E. Alonso, T. Mora, G. Niesi, and M. Raimondo. An algorithm for computing analytic branches of space curves at singular points. In *Proc. of the 1992 International Workshop on Mathematics Mechanization*, pages 135–166. International Academic Publishers, 1992.
- [2] E. Becker, T. Mora, M. G. Marinari, and C. Traverso. The shape of the shape lemma. In *Proc. of ISSAC'94*, pages 129–133. ACM Press, 1994.
- [3] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot. Representation for the radical of a finitely generated differential ideal. In *Proc. of ISSAC'95*, pages 158–166, 1995.
- [4] F. Boulier, F. Lemaire, and M. Moreno Maza. Well known theorems on triangular systems. Technical Report LIFL 2001–09, Université Lille I, LIFL, 2001.
- [5] C. Chen, J. H. Davenport, J. P. May, M. Moreno Maza, B. Xia, and R. Xiao. Triangular decomposition of semi-algebraic systems. *J. Symb. Comput.*, 49:3–26, 2013.
- [6] C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza, and W. Pan. Comprehensive triangular decomposition. In *Proc. of CASC'07*, volume 4770 of *Lecture Notes in Computer Science*, pages 73–101. Springer Verlag, 2007.
- [7] C. Chen and M. Moreno Maza. Algorithms for computing triangular decomposition of polynomial systems. *J. Symb. Comput.*, 47(6):610–642, 2012.
- [8] C. Chen and M. Moreno Maza. Algorithms for computing triangular decomposition of polynomial systems. *Journal of Symbolic Computation*, 47(6):610 – 642, 2012.
- [9] C. Chen, M. Moreno Maza, B. Xia, and L. Yang. Computing cylindrical algebraic decomposition via triangular decomposition. In *Proc. of ISSAC'09*, pages 95–102, 2009.
- [10] S. C. Chou and X. S. Gao. Computations with parametric equations. In *Proc. ISSAC'91*, pages 122–127, Bonn, Germany, 1991.
- [11] S. C. Chou and X. S. Gao. A zero structure theorem for differential parametric systems. *J. Symb. Comput.*, 16(6):585–596, 1993.
- [12] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms*. Springer-Verlag, 2nd edition, 1997.
- [13] J. Della Dora, C. Dicrescenzo, and D. Duval. About a new method for computing in algebraic number fields. In *Proc. of EUROCAL' 85 Vol. 2*, volume 204 of *Lect. Notes in Comp. Sci.*, pages 289–290. Springer-Verlag, 1985.

- [14] D. Dominique. Rational Puiseux expansions. *Compos. Math.*, 70(2):119–154, 1989.
- [15] G. Fischer. *Plane Algebraic Curves*. American Mathematical Society, United States of America, 2001.
- [16] X. S. Gao, J. Van der Hoeven, Y. Luo, and C. Yuan. Characteristic set method for differential-difference polynomial systems. *J. Symb. Comput.*, 44:1137–1163, 2009.
- [17] É. Hubert. Factorization free decomposition algorithms in differential algebra. *J. Symb. Comput.*, 29(4-5):641–662, 2000.
- [18] M. Kalkbrener. Algorithmic properties of polynomial rings. *J. Symb. Comput.*, 26(5):525–581, 1998.
- [19] F. Lemaire, M. Moreno Maza, and Y. Xie. The **RegularChains** library. In *Maple 10*, Maplesoft, Canada, 2005. Refereed software.
- [20] François Lemaire, Marc Moreno Maza, Wei Pan, and Yuzhen Xie. When does  $\langle t \rangle$  equal  $\text{sat}(t)$ ? *J. Symb. Comput.*, 46(12):1291–1305, 2011.
- [21] S. Marcus, M. Moreno Maza, and P. Vrbik. On Fulton’s algorithm for computing intersection multiplicities. In *Proc. of CASC’12*, pages 198–211, 2012.
- [22] J. Maurer. Puiseux expansion for space curves. *manuscripta mathematica*, 32:91–100, 1980.
- [23] D. Mumford. *The Red Book of Varieties and Schemes*. Springer, 2nd. edition, 1999.
- [24] J. R. Munkres. *Topology*. Prentic Hall, United States of America, 2nd. edition, 2000.
- [25] A. Parusinski and G. Rond. The Abhayankar-Jung theorem. *J. Algebra*, 365:29–41, 2012.
- [26] J. F. Ritt. *Differential Equations from an Algebraic Standpoint*, volume 14. American Mathematical Society, New York, 1932.
- [27] T. Shimoyama and K. Yokoyama. Localization and primary decomposition of polynomial ideals. *J. Symb. Comput.*, 22(3):247–277, 1996.
- [28] R. J. Walker. *Algebraic Curves*. Springer-Verlag, Berlin-New York, 1978.
- [29] D. K. Wang. The **Wsolve** package. <http://www.mmrc.iss.ac.cn/~dwang/wsolve.txt>.
- [30] D. M. Wang. Epsilon 0.618. <http://www-calfor.lip6.fr/~wang/epsilon>.
- [31] L. Yang, X. R. Hou, and B. Xia. A complete algorithm for automated discovering of a class of inequality-type theorems. *Science in China, Series F*, 44(6):33–49, 2001.